Clustering of inertial particles in a turbulent flow

Tov Elperin[1],* Nathan Kleeorin[1],† Michael A. Liberman[2],‡
Victor S. L'vov[3],§ Anna Pomyalov[3],¶ and Igor Rogachevskii[1]**

[1]The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, P. O. Box 653, Beer-Sheva 84105, Israel
[2]Department of Physics, Uppsala University, Box 530, SE-751 21, Uppsala, Sweden
[3]Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

(Dated: February 8, 2008)

We analyzed formation of small-scale inhomogeneities of particle spatial distribution (particle clustering) in a turbulent flow. The particle clustering is a consequence of a spontaneous breakdown of their homogeneous space distribution, and is caused by a combined effect of the particle inertia and a finite correlation time of the turbulent velocity field. Theory of the particle clustering is extended to the case when the particle Stokes time is larger than the Kolmogorov time scale, but is much smaller than the correlation time at the integral scale of turbulence. The criterion of the clustering instability is obtained. Applications of the analyzed effects to the dynamics of inertial particles in industrial turbulent flows are discussed.

PACS numbers: 47.27.Qb

I. INTRODUCTION

It is generally believed that turbulence promotes mixing (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9]). However, laboratory experiments and observations in the atmospheric turbulence show formation of long-living inhomogeneities in concentration distribution of small inertial particles and droplets in turbulent fluid flows (see, e.g., [10, 11, 12]). The origin of these inhomogeneities is not always clear but there influence on the mixing can be hardly overestimated.

The goal of this study is to analyze the particle-air interaction leading to the formation of strong inhomogeneities of particle distribution, referred to as *particle clustering*(see [13, 14], and references therein). Particle clustering is a consequence of a spontaneous breakdown of their homogeneous space distribution. As a result of the nonlinear stage of clustering, the local density of particles may rise by orders of magnitude and strongly increase the probability of particle-particle collisions.

It was suggested in [13, 15, 16] that the main reason for the particle clustering is their inertia: the particles inside the turbulent eddies are carried out to the boundary regions between the eddies by the inertial forces. This mechanism of the preferential concentration acts in all scales of turbulence, increasing toward small scales. Later, this was contested in Refs. [17, 18] using the so-called "Kraichnan model" [19] of turbulent advection by the delta-correlated in time random velocity field, whereby the clustering instability did not occur. However, it was shown in Ref. [14] that accounting for a finiteness of correlation time of the fluid velocity field results in the clustering instability of heavy particles. The theory of the clustering instability of the inertial particles advected by a turbulent velocity field was developed in Ref. [14] and applied to the dynamics of aerosols in the turbulent atmosphere.

In the present study we extended the theory of particle clustering to the case when the particle Stokes time is larger than the Kolmogorov time scale, but is much smaller than the correlation time at the integral scale of turbulence. The paper is organized as follows. In Sec. II we present a qualitative analysis of the clustering instability that causes formation of particle clusters in a turbulent flow. In Sec. III we evaluate the characteristic parameters that affect clusters formation as a result of the clustering instability: the particle response time τ_p, the Kolmogorov micro-scale time τ_η, the characteristic velocity of small and large particles in turbulent fluid; the degree of compressibility of the particle velocity field, $\sigma_v(a)$, where a is the particle size. In Sec. III B we study the velocity of small and large particles in the turbulent fluid, required for evaluation of the effect of turbulent diffusion in these two regimes. In Sec. IV we present a quantitative analysis for the clustering instability of the second moment of particle number density. This allows us to generalize the criterion of the clustering instability, obtained in Ref. [14]. Finally, in Sec. V we overview the nonlinear effects which lead to saturation of the clustering instability and determine the particle number density in the cluster. In Sec. V we also present numerical estimates for droplet dynamics for the conditions pertinent to turbulence in diesel engines.

*Electronic address: elperin@menix.bgu.ac.il; URL: http://www.bgu.ac.il/~elperin
†Electronic address: nat@menix.bgu.ac.il
‡Electronic address: Michael.Liberman@dec11.fysik.uu.se; URL: http://lili.fysik.uu.se/
§Electronic address: Victor.Lvov@Weizmann.ac.il; URL: http://lvov.weizmann.ac.il
¶Electronic address: Anna.Pomyalov@Weizmann.ac.il
**Electronic address: gary@menix.bgu.ac.il; URL: http://www.bgu.ac.il/~gary

II. CLUSTERING OF PARTICLES IN TURBULENT GAS

A. Basic Equations

To analyze dynamics of particles we use the standard continuous media approximation, introducing the number density $n(t,\mathbf{r})$ of spherical particles with radius a.

The particles are advected by a turbulent velocity field $\mathbf{u}(t,\mathbf{r})$. Since typically the velocity of the carrier gas is much smaller than the sound velocity, the incompressibility constrain $\operatorname{div}\mathbf{u}(t,\mathbf{r}) = 0$ is applicable. The particle material density ρ_p is much larger than the density ρ of the ambient fluid. For heavy particles $\mathbf{v}(t,\mathbf{r}) \neq \mathbf{u}(t,\mathbf{r})$ due to the particle inertia [20, 21, 22]. Therefore, the compressibility of the particle velocity field $\mathbf{v}(t,\mathbf{r})$ must be taken into account (see [13, 14, 15, 16]), since the growth rate of the clustering instability, γ, is proportional to $\langle|\operatorname{div}\mathbf{v}(t,\mathbf{r})|^2\rangle$, where $\langle\cdot\rangle$ denotes ensemble average.

Let $\Theta(t,\mathbf{r})$ be the deviation of the particle number density $n(t,\mathbf{r})$ from its uniform mean value $\bar{n} \equiv \langle n\rangle$:

$$\Theta(t,\mathbf{r}) = n(t,\mathbf{r}) - \bar{n}\,. \tag{2.1}$$

The pair correlation function of $\Theta(t,\mathbf{r})$ is defined as

$$\Phi(\mathbf{R},\mathbf{r},t) \equiv \langle\Theta(t,\mathbf{r}+\mathbf{R})\Theta(t,\mathbf{r})\rangle\ . \tag{2.2}$$

For the sake of simplicity we will consider only a spatially homogeneous, isotropic case when $\Phi(\mathbf{R},\mathbf{r},t)$ depends only on the separation distance R and time t:

$$\Phi(t,\mathbf{R},\mathbf{r}) \to \Phi(t,R)\ . \tag{2.3}$$

Denote the probability of the pair collisions between particles as $p(t)$ which can be expressed as

$$p(t) \propto c\left\{1+\frac{\langle[n(t,\mathbf{r})]^2\rangle}{\bar{n}^2}\right\} = c\left[1+\frac{\Phi(t,0)}{\bar{n}^2}\right]\ . \tag{2.4}$$

Obviously, a large increase of $\Phi(t,R)$ above the level of $\bar{n}^2$ leads to a strong grows in the frequency of the particle collisions.

In the analytical treatment of the problem we use the standard equation for $n(t,\mathbf{r})$:

$$\frac{\partial n(t,\mathbf{r})}{\partial t} + \nabla\cdot[n(t,\mathbf{r})\mathbf{v}(t,\mathbf{r})] = D\triangle n(t,\mathbf{r}), \tag{2.5}$$

where D is the coefficient of molecular (Brownian) diffusion. The equation for $\Theta(t,\mathbf{r})$ follows from Eq. (2.5):

$$\begin{aligned}\frac{\partial\Theta(t,\mathbf{r})}{\partial t} &+ [\mathbf{v}(t,\mathbf{r})\cdot\nabla]\Theta(t,\mathbf{r}) \\ &= -\Theta(t,\mathbf{r})\operatorname{div}\mathbf{v}(t,\mathbf{r}) + D\triangle\Theta(t,\mathbf{r})\ .\end{aligned} \tag{2.6}$$

Here we neglected the term $\propto \bar{n}\operatorname{div}\mathbf{v}$, describing the effect of an external source of fluctuations. It was shown in Ref. [14] that this effect is usually much smaller than the effect of self-excitation of fluctuations of particle number density.

One can use Eq. (2.6) to derive equation for $\Phi(t,R)$ by averaging the equation for $\Theta(t,\mathbf{r}+\mathbf{R})\Theta(t,\mathbf{r})$ over statistics of the advected turbulent velocity field $\mathbf{v}(t,\mathbf{r})$. In general this procedure is quite involved even for simple models of the advecting velocity fields, see, e.g., Ref. [14]. Nevertheless, the qualitative understanding of the underlying physics of the clustering instability, leading to both, the exponential growth of $\Phi(t,R)$ and its nonlinear saturation, can be elucidated by a more simple and transparent analysis, that is presented below.

B. Qualitative analysis of the clustering instability

1. *On Richardson-Kolmogorov cascade theory of turbulence*

In our discussion we will use the well known Richardson-Kolmogorov cascade theory of turbulence (see, e.g., Refs. [1, 23, 24]). For the large Reynolds numbers $\mathcal{R}e \gg 1$ the characteristic scale L of energy injection (outer scale) is much larger than the length of the dissipation scales (*viscous scale* η) $L \gg \eta$. In the so-called *inertial interval* of scales, where $L > r > \eta$, the statistics of turbulence within the Kolmogorov theory is governed by the only dimensional parameter, ε, the rate of the turbulent energy dissipation. Then, the velocity $u(r)$ and the energy of turbulent motion $E(r)$ at the characteristic scale r (referred below as r-eddies) may be found by the dimensional reasoning:

$$u(r) \approx (\varepsilon r)^{1/3}, \quad E(r) = \frac{1}{2}\rho[u(r)]^2. \tag{2.7}$$

Similarly, the turnover time of r-eddies, $\tau(r)$, which is of the order of their life time, may be estimated as

$$\tau(r) \approx r/u(r) \approx \varepsilon^{-1/3}r^{2/3}. \tag{2.8}$$

To elucidate the clustering instability let us consider a cluster of particles with a characteristic scale ℓ moving with the velocity $\mathbf{V}_{\rm cl}(t)$. The scale ℓ is a parameter which governs the growth rate of the clustering instability, γ. It sets the bounds for two distinct intervals of scales: $L > r > \ell$ and $\ell > r > \eta$. However, if the size of particles a is larger than the viscous scale η, the second range of scales becomes $\ell > r > a$, because we cannot consider scales which are smaller than the size of particles. Large r-eddies with $r > \ell$ sweep the ℓ-cluster as a whole and determine the value of $\mathbf{V}_{\rm cl}(t)$. This results in the diffusion of the clusters, and eventually affects their distribution in a turbulent flow. The small r-eddies determine the dynamics of particles inside the cluster. The role of these eddies is multifold. First, they lead to the turbulent diffusion of the particles within the scale of a cluster size. Second, due to the particle inertia they tend to accumulate particles in the regions with small vorticity, which leads to the preferential concentration of the particles. Third, the particle inertia also causes a transport of fluctuations of particle number density from smaller scales

to larger scales, i.e., in regions with larger turbulent diffusion. The latter can decrease the growth rate of the clustering instability (see below). Thus, the clustering is determined by the competition between these three processes.

2. Effect of turbulent diffusion

We consider Eq. (2.6) for $\Theta(t,\mathbf{r})$ in the coordinate system co-moving with the ℓ-cluster and assume $\mathbf{v}(t,\mathbf{r}) \approx \mathbf{u}(t,\mathbf{r})$. In this reference frame the advectived velocity $\mathbf{v}(t,\mathbf{r})$ should be replaced by $[\mathbf{v}(t,\mathbf{r})-\mathbf{V}_{\rm cl}(t)]$. In particular, the advection term in Eq. (2.6) takes the form

$$\mathrm{Adv} \equiv \{[\mathbf{v}(t,\mathbf{r})-\mathbf{V}_{\rm cl}(t)]\cdot\nabla\}\,\Theta(t,\mathbf{r}) \ . \tag{2.9}$$

Averaging this term over statistics of turbulent velocity field leads to the turbulent diffusion of particles. It is well known (see, e.g. Ref. [1]) that the turbulent diffusion can be modelled by the renormalization of the diffusion term in the right hand side (RHS) of Eq. (2.6):

$$D \Rightarrow D + D_{\rm T}, \tag{2.10}$$

where $D_{\rm T}$ is the turbulent diffusion coefficient. The main contribution to the advected velocity in Eq. (2.9) is due to the velocity of ℓ-eddies, $v(\ell)$. Therefore $D_{\rm T}$ becomes a function of scale ℓ, $D_{\rm T} \Rightarrow D_{\rm T}(\ell)$, and can be estimated from the parameters of ℓ-eddies, using dimensional reasoning:

$$D_{\rm T}(\ell) \approx \frac{1}{3}\ell v(\ell) \ . \tag{2.11}$$

Here we used the commonly accepted [23, 24] numerical factor $\frac{1}{3}$ for the turbulent diffusion coefficient in an isotropic turbulence. Now the part of Eq. (2.6) describing the turbulent diffusion may be written as

$$\frac{\partial\Theta(t,\mathbf{r})}{\partial t} = -D_{\rm T}(\ell)\triangle\Theta(t,\mathbf{r}) \ . \tag{2.12}$$

During the linear stage of the cluster evolution the particle distribution inside the cluster does not change. Therefore, the function $\Theta(t,\mathbf{r})$ in the expression for the correlation function Φ can be factorized:

$$\Theta(t,\mathbf{r}) \Rightarrow A_\ell(t)\theta_\ell(r) \ . \tag{2.13}$$

Then, Eq. (2.12) yields

$$A_\ell^2(t) = A_\ell^2(0)\exp[-\gamma_{\rm dif}\,t]\ ,\ \gamma_{\rm dif} = \frac{2D_{\rm T}(\ell)}{\ell^2} \approx \frac{2v(\ell)}{3\ell}\ , \tag{2.14}$$

where the Laplace operator is evaluated as $1/\ell^2$ and we used Eq. (2.11).

3. Effect of particles inertia

For the qualitative analysis of the particle inertia we consider Eq. (2.6), written in the co-moving reference frame, taking into account only inertia term:

$$\frac{\partial\Theta(t,\mathbf{r})}{\partial t} = -\Theta(t,\mathbf{r})\,\mathrm{div}[\mathbf{v}(t,\mathbf{r})-\mathbf{V}_{\rm cl}(t)] \ . \tag{2.15}$$

As before we can factorize the function $\Theta(t,\mathbf{r})$ according to Eq. (2.13) and simplify the partial differential equation (2.15), reducing it to the differential equation for the cluster amplitude $A_\ell(t)$:

$$\frac{dA_\ell(t)}{dt} = -A_\ell(t)\,b_\ell(t) \ . \tag{2.16}$$

Here we neglected the $\mathbf{r}$-dependence of the divergence term inside the cluster in the RHS of Eq. (2.15), so that this term becomes a function of t only:

$$\mathrm{div}[\mathbf{v}(t,\mathbf{r})-\mathbf{V}_{\rm cl}(t)] \Rightarrow b_\ell(t) \ . \tag{2.17}$$

In the turbulent flow field the function $b_\ell(t)$ may be considered as a random process with some correlation time τ_b, which will be evaluated below. Since the instability of ℓ-clusters is caused by ℓ-eddies, the correlation time τ_b can be estimated as the turnover time of the particle velocity field of ℓ-eddies:

$$\tau_b \approx \ell/v(\ell) \ . \tag{2.18}$$

The evaluation of the mean square value of $b_\ell(t)$ requires a more careful consideration. One cannot estimate $\mathrm{div}\,\mathbf{v}(\mathbf{r})$ by the dimensional reasoning as $v(r)/r$. Indeed, in the incompressible flow $\mathrm{div}\,\mathbf{v}(\mathbf{r})=0$. To elucidate this issue we introduce a dimensionless parameter σ_v, a *degree of compressibility* of the velocity field of particles in ℓ-clusters, $\mathbf{v}_\ell(t,\mathbf{r})$, defined by

$$\sigma_v \equiv \langle[\mathrm{div}\,\mathbf{v}_\ell]^2\rangle/\langle|\nabla\times\mathbf{v}_\ell|^2\rangle \ . \tag{2.19}$$

This parameter may be of the order of 1 (see Refs. [15, 17]). At the moment we can evaluate $b_\ell(t)$ via this yet unknown parameter σ_v as follows:

$$\langle b_\ell(t)\rangle = 0, \quad \langle b_\ell^2(t)\rangle \approx \sigma_v[v(\ell)/\ell]^2 \ . \tag{2.20}$$

Let us show that the stochastic differential equation (2.16) results in an exponential growth of $\langle A_\ell^2(t)\rangle$, i.e. in the instability. Indeed, the solution of this equation reads:

$$A_\ell(t) = A_\ell(0)\exp[-I(t)]\ , \quad I(t) \equiv \int_0^t b_\ell(\tau)d\tau \ . \tag{2.21}$$

Integral $I(t)$ may be written as the sum of integrals I_n over small time intervals τ_b:

$$I(t) = \sum_{n=1}^{N} I_n\ , \quad I_n \equiv \int_{n\tau_b}^{(n+1)\tau_b} b_\ell(\tau)d\tau, \quad N = \frac{t}{\tau_b} \ . \tag{2.22}$$

By definition, τ_b is the correlation time of the random process $b_\ell(t)$. Therefore the integrals I_n can be considered as independent random variables. According to the central limit theorem, the sum of a large number of statistically independent random variables is distributed as the Gaussian random variable. Therefore the total integral $I(t)$ can be estimated as

$$I(t) \simeq \sqrt{N\langle I_n^2\rangle}\,\tilde{\zeta}\,, \quad \langle I_n^2\rangle = \tau_b^2\langle b_\ell^2(t)\rangle\ . \tag{2.23}$$

Here $\tilde{\zeta}$ is a Gaussian random variable with zero mean and unit variance. The probability density function of $\tilde{\zeta}$ reads

$$\mathcal{P}(\tilde{\zeta}) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{\zeta}^2}{2}\right). \tag{2.24}$$

Using Eqs. (2.21), (2.24) and (2.22) we obtain

$$\begin{aligned}\langle A_\ell^2(t)\rangle &\simeq A_\ell^2(0)\int\exp[-2\sqrt{N\langle I_n^2\rangle}\tilde{\zeta}]\mathcal{P}(\tilde{\zeta})d\tilde{\zeta}\\ &= A_\ell^2(0)\exp[\gamma_{in}t]\,,\end{aligned} \tag{2.25}$$

where the growth rate γ_{in} is given by

$$\gamma_{\rm in} = \frac{N\langle I_n^2\rangle}{\tau_b} \approx 2\tau_b\langle b_\ell^2(t)\rangle \approx \frac{2\sigma_v\, v(\ell)}{\ell}\ . \tag{2.26}$$

In the last estimate we used Eqs. (2.18) and (2.20). It is clearly seen that the source of the instability is a nonzero value of σ_v, i.e. a *compressibility of the particle velocity field*, $\mathbf{v}(t,\mathbf{r})$.

In the case of the small enough particles, $\tau_p \le \tau(\eta)$, where $\tau(\eta)$ is the turnover time of the smallest, Kolmogorov micro-scale eddies. In this case all the particles are almost fully involved in turbulent motion, and one concludes that $u(t,\mathbf{r}) \approx v(t,\mathbf{r})$ and $v(\ell) \approx u(\ell)$. Also we will see below that in this case the largest value of σ_v is attained for $\ell \approx \eta$. Hence for $\tau_p \le \tau(\eta)$ the most unstable clusters are those with $\ell \approx \eta$. In summary:

• In the case $\tau_p \le \tau(\eta)$, the characteristic scale of the most unstable clusters of small enough particles is of the order of Kolmogorov micro-scale of turbulence, η.

• The characteristic growth rate of the clustering instability is of the order of the turnover frequency of η-eddies, $1/\tau(\eta)$.

• The particle clusters are unstable for heavy enough particles, such that the degree of compressibility $\sigma_v(\eta)$ of their effective advective velocity field exceeds some threshold value $\sigma_{\rm cr} \approx 0.3$ (see Ref. [14]).

III. THE PARTICLE VELOCITY FIELD

In this section we discuss the compressibility of the particle velocity field. In particular, to clarify the range of validity of the Stokes approximation for the particle motion in fluid, in Sec. III A we evaluate the particle response time τ_p as compared with the Kolmogorov micro-scale time τ_η. In Sec. III B we study the velocity of small and large particles in the turbulent fluid, required for evaluation of the effect of turbulent diffusion in these two regimes. This study also allows us to find the dependence of the degree of compressibility of the particle velocity field, σ_v, on the particle response time.

A. Characteristic time scales and validly of the Stokes approximation

Let us assume that the radius of the particles is small, so that the particle Reynolds number $\mathcal{R}e_p$ is smaller than the critical value, $\mathcal{R}e_{\rm cr}$, at which the laminar flow over a particle looses its stability. Then, we can apply the Stokes approximation. It states that the fluid-particle friction force is proportional to the *slip velocity*, the difference between the particle velocity and the fluid velocity. A careful analysis performed by Lumley [25] shows that in a turbulent flow the validity condition for $\mathcal{R}e_p \le \mathcal{R}e_{\rm cr}$ may be expressed via the particle radius a and the Kolmogorov micro-scale η as follows:

$$a \le 2\eta(\rho/\rho_p)^{1/3}. \tag{3.1}$$

In the present analysis the ratio of the inertial time scale of the particles (the Stokes time scale τ_p) and the turnover time of η-eddies in the Kolmogorov micro-scale $\tau(\eta)$, is of primary importance. The particle response time is given by

$$\tau_p = \frac{m_p}{6\pi\nu\rho a} = \frac{2\rho_p a^2}{9\rho\nu}\,, \tag{3.2}$$

where the particle mass m_p is:

$$m_p = \frac{4\pi}{3}a^3\rho_p\ . \tag{3.3}$$

The Kolmogorov micro-scale η is determined from the condition that the Reynolds number for eddies of scale η is equal to 1:

$$\mathcal{R}e_\eta = \eta v(\eta)/\nu = 1\ . \tag{3.4}$$

Here $v(\eta)$ is the characteristic velocity of η-scale eddies. It is related to the turnover time of the eddies as $\tau(\eta) = \eta/v(\eta)$. This allows us to rewrite the expression (3.4) as follows:

$$\tau(\eta) = \eta^2/\nu. \tag{3.5}$$

Then, the ratio of the time-scales τ_p and $\tau(\eta)$ follows from Eqs. (3.2) and (3.5):

$$\frac{\tau_p}{\tau(\eta)} = \frac{2\rho_p a^2}{9\rho\eta^2}\ . \tag{3.6}$$

Substituting the condition (3.1) for the validity of the Stokes approximation we find

$$\frac{\tau_p}{\tau(\eta)} \le \left(\frac{\rho_p}{\rho}\right)^{1/3}\,, \tag{3.7}$$

Equation (3.7) implies that for "heavy" particles in a gas, that satisfy the Stokes approximation, the particle response time scale may be about ten times larger than the Kolmogorov time scale: $\tau_p \leq 10\tau(\eta)$.

B. Particle velocity in turbulent fluid

1. Equation for a particle velocity

Assuming that the particles are small enough such that Eq. (3.1) is valid, we can apply Stokes' law for the fluid-particle friction force $\mathbf{F}_p(t,\mathbf{r})$:

$$\mathbf{F}_p(t,\mathbf{r}) = \zeta[\mathbf{v}_p(t,\mathbf{r}) - \mathbf{u}(t,\mathbf{r})]\,, \tag{3.8}$$

with the Stokes friction coefficient ζ given by

$$\zeta = 6\,\pi\,\rho\nu\,a\ . \tag{3.9}$$

The Newton equation for a particle reads:

$$m_p \frac{d\mathbf{v}_p(t,\mathbf{r})}{dt} = -\mathbf{F}_p(t,\mathbf{r}) = \zeta[\mathbf{u}(t,\mathbf{r}) - \mathbf{v}_p(t,\mathbf{r})]\ . \tag{3.10}$$

Here the total time derivative (d/dt) takes into account the time dependence of the particle coordinate $\mathbf{r}$:

$$\frac{d}{dt} = \left[\frac{\partial}{\partial t} + \mathbf{v}_p(t,\mathbf{r})\cdot\nabla\right]\ . \tag{3.11}$$

Now Eq. (3.10) takes the form:

$$\left\{\tau_p\left[\frac{\partial}{\partial t} + \mathbf{v}_p(t,\mathbf{r})\cdot\nabla\right] + 1\right\}\mathbf{v}_p(t,\mathbf{r}) = \mathbf{u}(t,\mathbf{r})\ . \tag{3.12}$$

In the following we analyze this equation in two limiting cases: for small particles with τ_p smaller than the turnover time of ℓ-eddies (Sec. III B 2), and for large particles, in Sec. III B 3.

2. Velocity of small particles

In this section we consider small particles for which the Stokes time τ_p is smaller than the turnover time of ℓ-eddies, $\tau(\ell) \approx \ell/u(\ell)$. These particles are completely involved in the motion of ℓ-eddies and for $\tau_p/\tau(\ell) = 0$, we can write $\mathbf{v}_\ell(t,\mathbf{r}) = \mathbf{u}_\ell(t,\mathbf{r})$. Here subscript "$\ell$" denotes that we are dealing with the velocity field of ℓ-eddies, $\mathbf{u}_\ell(t,\mathbf{r})$, and the velocity of particles in ℓ-clusters, $\mathbf{v}_\ell(t,\mathbf{r})$, that was generated by ℓ-eddies. In this approximation the velocity field $\mathbf{v}_\ell(t,\mathbf{r})$ is incompressible. Therefore, the compressibility parameter σ_v may be determined by the first order corrections.

To find the corrections to $\mathbf{v}_\ell(t,\mathbf{r})$ linear in τ_p, we consider a formal solution of Eq. (3.12) in the form:

$$\mathbf{v}_\ell(t,\mathbf{r}) = \left[\tau_p\frac{d}{dt} + 1\right]^{-1}\mathbf{u}_\ell(t,\mathbf{r})\ . \tag{3.13}$$

Here the inverse operator is understood as its Taylor series expansion:

$$\mathbf{v}_\ell(t,\mathbf{r}) = \mathbf{u}_\ell(t,\mathbf{r}) - \tau_p\frac{d\mathbf{u}_\ell(t,\mathbf{r})}{dt} + \frac{\tau_p^2}{2}\left[\frac{d\mathbf{u}_\ell(t,\mathbf{r})}{dt}\right]^2 + ..., \tag{3.14}$$

In the linear in τ_p approximation, this solution becomes

$$\mathbf{v}_\ell(t,\mathbf{r}) = \mathbf{u}_\ell(t,\mathbf{r}) - \tau_p\left[\frac{\partial}{\partial t} + \mathbf{u}_\ell(t,\mathbf{r})\cdot\nabla\right]\mathbf{u}_\ell(t,\mathbf{r})\,, \tag{3.15}$$

where in the RHS we replaced $\mathbf{v}_\ell(t,\mathbf{r}) \to \mathbf{u}_\ell(t,\mathbf{r})$. Consequently, in this approximation

$$\mathrm{div}\,\mathbf{u}_\ell(t,\mathbf{r}) = \frac{du_\ell^\alpha(t,\mathbf{r})}{dr^\alpha} \approx \tau_p\frac{du_\ell^\beta(t,\mathbf{r})}{dr^\alpha}\frac{du_\ell^\alpha(t,\mathbf{r})}{dr^\beta} \tag{3.16}$$

(see Ref. [13]). Equation (3.16) allows to determine the compressibility parameter σ_v, (2.19), as follows:

$$\sigma_v \approx \left[\frac{\tau_p}{\tau(\ell)}\right]^2 \approx \left(\frac{2\rho_p}{9\rho}\right)^2\left(\frac{a}{\eta}\right)^4\left(\frac{\eta}{\ell}\right)^{4/3}\ . \tag{3.17}$$

In deriving this equation we estimated $[du_\ell^\alpha(t,\mathbf{r})/dr^\alpha]$ as $[1/\tau(\ell)]$ and used Eqs. (3.6) and (2.8).

Let a_* be the characteristic size of particles for which $\tau_p = \tau(\eta)$ for $\ell = \eta$ and, respectively, $\sigma_v = 1$:

$$a_* \equiv \eta\sqrt{9\rho/2\rho_p}\ . \tag{3.18}$$

Using this notation Eq. (3.17) can be rewritten as

$$\frac{\tau(\ell)}{\tau_p} \approx \left(\frac{a_*}{a}\right)^2\left(\frac{\ell}{\eta}\right)^{2/3}, \tag{3.19}$$

$$\sigma_v \approx \left(\frac{a}{a_*}\right)^4\left(\frac{\eta}{\ell}\right)^{4/3}\ . \tag{3.20}$$

Equations (3.17) and (3.20) are valid only if $\sigma_v < 1$, otherwise the approximation of small $\tau_p/\tau(\ell)$ ratio is violated.

3. Velocity of large particles

a. Effective equation of motion. In this section we consider the opposite case: the large particles with τ_p is larger than the turnover time of the smallest eddies in the Kolmogorov microscale $\tau(\eta)$, but smaller than the turnover time of the largest eddies $\tau(L)$. Denote by ℓ_* the characteristic scale of eddies for which

$$\tau_p = \tau(\ell_*)\ . \tag{3.21}$$

This scale as well as the particles cluster size was introduced in Ref. [13]. The eddies with $\ell \gg \ell_*$ almost fully involve particles in their motions, while the eddies with $\ell \ll \ell_*$ do not affect the particle motions within the zero order approximation in the ratio $[\tau(\ell)/\tau_p] \ll 1$.

To determine $\boldsymbol{v}_\ell(t,\boldsymbol{r})$ we consider Eq. (3.12) in the co-moving with ℓ-eddies frame, where the surrounding fluid velocity $\boldsymbol{u}$ equals to the relative velocity of the ℓ-eddy at $\boldsymbol{r}$, i.e., $\boldsymbol{u}(t,\boldsymbol{r}) \Rightarrow \boldsymbol{u}_\ell(t,\boldsymbol{r})$. At this point one has to take into account that the ℓ-eddy is swept out by all ℓ'-eddies of larger scales, $\ell' > \ell$, while the particle participates in motions of ℓ'-eddies with $\ell' > \ell_* > \ell$. Therefore, the relative velocity U_ℓ of the ℓ-eddy and the particle is determined by ℓ'-eddies with the intermediate scales, $\ell_* > \ell' > \ell$. This velocity is determined by the contribution of ℓ_*-eddies, and can be considered as a time and space independent constant $\boldsymbol{u}_*$ during the life time of the ℓ eddy and inside it. Velocity $\boldsymbol{u}_*$ in our approach is random and we have to average the final result over statistics of ℓ_*-eddies. Then Eq. (3.12) becomes

$$\left(\tau_p\frac{\partial}{\partial t}+1\right)\boldsymbol{v}_\ell(t,\boldsymbol{r}) = \boldsymbol{u}_\ell(t,\boldsymbol{r}+\boldsymbol{u}_* t) -\tau_p\left[\boldsymbol{v}_\ell(t,\boldsymbol{r})\cdot\nabla\right]\boldsymbol{v}_\ell(t,\boldsymbol{r}) \ . \tag{3.22}$$

In Eq. (3.22) the velocity $\boldsymbol{u}_\ell$ is calculated at point $\boldsymbol{r}$ and the velocity $\boldsymbol{v}_\ell$ is at $\boldsymbol{r}-\boldsymbol{u}_*\,t$. For the sake of convenience we redefine here $\boldsymbol{r}-\boldsymbol{u}_*\,t \equiv \boldsymbol{r}' \Rightarrow \boldsymbol{r}$ and, respectively, $\boldsymbol{r}=\boldsymbol{r}'+\boldsymbol{u}_*\,t \Rightarrow \boldsymbol{r}+\boldsymbol{u}_*\,t$.

b. First non-vanishing contribution to v_ℓ. Clearly, $v_\ell(t,\boldsymbol{r}) \ll u_\ell$ for $\ell \ll \ell_*$. Therefore we can find the first non-vanishing contribution to $\boldsymbol{v}_\ell(t,\boldsymbol{r})$ in the limit $[\tau(\ell)\ll\tau_p]$ by considering the linear version of Eq. (3.22):

$$\left(\tau_p\frac{\partial}{\partial t}+1\right)\boldsymbol{v}_\ell(t,\boldsymbol{r}) = \boldsymbol{u}_\ell(t,\boldsymbol{r}+\boldsymbol{u}_* t) \ . \tag{3.23}$$

In the $\omega,\boldsymbol{k}$ representation this equation takes the form:

$$(i\omega\,\tau_p+1)\,\boldsymbol{v}_\ell(\omega,\boldsymbol{k}) = \boldsymbol{u}_\ell(\omega-\boldsymbol{k}\cdot\boldsymbol{u}_*,\boldsymbol{k})\,, \tag{3.24}$$

that allows one to find the relationship between the second order correlation functions $F^{\alpha\beta}_{v,\ell}(\omega,\boldsymbol{k})$ and $F^{\alpha\beta}_{u,\ell}(\omega,\boldsymbol{k})$ of the velocity fields $\boldsymbol{v}_\ell$ and $\boldsymbol{u}_\ell$:

$$F^{\alpha\beta}_{v,\ell}(\omega,\boldsymbol{k}) = \frac{1}{\omega^2\tau_p^2+1}F^{\alpha\beta}_{u,\ell}(\omega-\boldsymbol{k}\cdot\boldsymbol{u}_*,\boldsymbol{k}) \ . \tag{3.25}$$

Functions $F^{\alpha\beta}_{u,\ell}(\omega,\boldsymbol{k})$ and $F^{\alpha\beta}_{v,\ell}(\omega,\boldsymbol{k})$ are defined as usual:

$$(2\pi)^4\delta(\omega+\omega')\,\delta(\boldsymbol{k}+\boldsymbol{k}')F^{\alpha\beta}_{u,\ell}(\omega,\boldsymbol{k}) \equiv \left\langle v^\alpha_\ell(\omega,\boldsymbol{k})\,v^\beta_\ell(\omega',\boldsymbol{k}')\right\rangle\,, \qquad \text{etc.} \tag{3.26}$$

The simultaneous correlation functions are related to their ω-dependent counterparts via the integral $\int d\omega/2\pi$, e.g.,

$$F^{\alpha\beta}_{v,\ell}(\boldsymbol{k}) = \int\frac{d\omega}{2\pi}F^{\alpha\beta}_{v,\ell}(\omega,\boldsymbol{k}) \ . \tag{3.27}$$

The tensorial structure of $F^{\alpha\beta}_{u,\ell}(\boldsymbol{k})$ follows from the incompressibility condition and the assumption of isotropy:

$$F^{\alpha\beta}_{u,\ell}(\boldsymbol{k}) = P^{\alpha\beta}(\boldsymbol{k})F_{u,\ell}(k)\,, \tag{3.28}$$

where $P^{\alpha\beta}(\boldsymbol{k})$ is the transversal projector:

$$P^{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta}-k^\alpha k^\beta/k^2 \ . \tag{3.29}$$

In the inertial range of scales the function $F^{\alpha\beta}_{u,\ell}(\omega,\boldsymbol{k})$ may be written in the following form:

$$F^{\alpha\beta}_{u,\ell}(\omega,\boldsymbol{k}) = P^{\alpha\beta}(\boldsymbol{k})\,F_{u,\ell}(k)\,\tau(\ell)\,f[\omega\,\tau(\ell)] \ . \tag{3.30}$$

Here the dimensionless function $f(x)$ is normalized as follows:

$$\int_\infty^\infty f(x)\,dx = 2\pi \ . \tag{3.31}$$

Now we can average Eq. (3.25) over the statistics of ℓ_*-eddies. Denoting the mean value of some function $g(x)$ as $\overline{g(x)}$ we have:

$$\begin{aligned}\overline{f[(\omega-\boldsymbol{k}\cdot\boldsymbol{u}_*)\,\tau(\ell)]} &\approx \overline{\delta[(\omega-\boldsymbol{k}\cdot\boldsymbol{u}_*)\,\tau(\ell)]} \\ &\approx \frac{\ell}{\tau(\ell)\,u_*}\,f_*\left(\frac{u_*\,\omega}{\ell}\right) \ .\end{aligned} \tag{3.32}$$

Here the dimensionless function $f_*(x)$ has one maximum at $x=0$, and it is normalized according to Eq. (3.31). The particular form of $f_*(x)$ depends on the statistics of ℓ_*-eddies and our qualitative analysis is not sensitive to this form. Thus, we may choose, for instance:

$$f_*(x) = 2/[x^2+1] \ . \tag{3.33}$$

In Eqs. (3.32) we took into account that the characteristic Doppler frequency of ℓ-eddies (in the random velocity field u_* of ℓ_*-eddies) may be evaluated as:

$$\gamma_{\rm D}(\ell) \equiv \sqrt{\overline{(\boldsymbol{k}\cdot\boldsymbol{u}_*)^2}} \simeq u_*/\ell \ . \tag{3.34}$$

This frequency is much larger than the characteristic frequency width of the function $f[\omega\tau(\ell)]$ (equal to $1/\tau(\ell)$), and therefore the function $f(x)$ in Eq. (3.32) may be approximated by the delta function $\delta(x)$.

After averaging, Eq. (3.25) may be written as

$$F^{\alpha\beta}_{v,\ell}(\omega,\boldsymbol{k}) = \frac{P^{\alpha\beta}(\boldsymbol{k})f_*(0)}{\omega^2\tau_p^2+1}\frac{\ell}{u_*}F_{u,\ell}(k) \ . \tag{3.35}$$

Here we took into account that $\tau_p \gg \ell/u_*$ that allows us to neglect the frequency dependence of $f_*(u_*\,\omega/\ell)$ and to calculate this function at $\omega=0$. Together with Eq. (3.35) this yields

$$F^{\alpha\beta}_{v,\ell}(\boldsymbol{k}) = P^{\alpha\beta}(\boldsymbol{k})\,F_{u,\ell}(k)\frac{\ell}{\tau_p\,u_*}\,, \tag{3.36}$$

where we used the estimate $f_*(0)\approx 2$, that follows from Eq. (3.33).

The equation (3.36) provides the relationship between the mean square relative velocity of ℓ-separated particles, v_ℓ, and the velocity of ℓ-eddies, u_ℓ:

$$v_\ell \simeq u_\ell\sqrt{\frac{\ell}{\tau_p\,u_*}} \simeq u_\ell\sqrt{\frac{\ell}{\ell_*}} \ . \tag{3.37}$$

c. Effective nonlinear equation. For a qualitative analysis of the role of the nonlinearity of the particle behavior in an ℓ-cluster we evaluate ∇ in the nonlinear term, Eq. (3.22), as $1/\ell$, neglecting the spatial dependence and the vector structure. The resulting equation in ω-representation reads:

$$\begin{aligned}(i\omega &+ \gamma_p)V_\ell(\omega) = \gamma_{\rm dr}U_\ell(\omega) + \mathcal{N}_\omega\,, \quad \gamma_p = 1/\tau_p\,,\\ \mathcal{N}_\omega &= -\frac{1}{2\pi\ell}\int d\omega_1 d\omega_2 \delta(\omega+\omega_1+\omega_2)V_\ell(\omega_1)V_\ell(\omega_2)\,,\\ V_\ell(\omega) &= \int v_\ell(t)\exp[-i\omega t]dt\,, \qquad (3.38)\\ v_\ell(t) &= \frac{1}{2\pi}\int V_\ell(\omega)\exp[i\omega t]d\omega\,.\end{aligned}$$

In the zeroth order (linear) approximation ($\mathcal{N}_\omega \to 0$)

$$V_\ell^{(0)}(\omega) = \frac{\gamma_p U_\ell(\omega)}{i\omega + \gamma_p}\,, \qquad (3.39)$$

which is the simplified version of Eq. (3.24). This allows us to find in the linear approximation

$$\langle v_\ell^2(t)\rangle = \int \frac{d\omega}{2\pi}\mathcal{F}_\ell(\omega) = \int \frac{d\omega}{2\pi}\frac{\gamma_p^2\overline{F_{u,\ell}(\omega)}}{\omega^2+\gamma_p^2}\,, \qquad (3.40)$$

where $F_{u,\ell}(\omega)$ is the correlation functions of $U_\ell(\omega)$:

$$2\pi\delta(\omega+\omega')F_{u,\ell}(\omega) = \langle U_\ell(\omega)U_\ell(\omega')\rangle\,, \qquad (3.41)$$

similarly to Eq. (3.26).

In the limit $\tau_p \gg \ell/u_*$ one can neglect in Eq. (3.40) the ω-dependence of $\overline{F_{u,\ell}(\omega)}$, which has characteristic width ℓ/u_* and conclude:

$$\begin{aligned} v_{\ell,0}^2 &\equiv \langle v_\ell^2(t)\rangle \approx \frac{\gamma_p}{2}\overline{F_{u,\ell}(0)} \approx u_\ell^2\frac{\ell\,\gamma_p}{u_*} \approx u_\ell^2\frac{\ell}{\ell_*},\\ u_\ell^2 &\equiv \langle u_\ell^2(t)\rangle\,, \qquad (3.42)\end{aligned}$$

in agreement with Eq. (3.37).

d. First nonlinear correction. To evaluate the first nonlinear correction to Eq. (3.42) one has to substitute $V_\ell(\omega)$ from Eq. (3.39) into Eq. (3.38) for $\mathcal{N}_\omega$:

$$\begin{aligned} V_{\ell,1}(\omega) &= -\frac{\gamma_p^2}{2\pi\ell}\int d\omega_1 d\omega_2\delta(\omega+\omega_1+\omega_2)\\ &\times \frac{U_\ell(\omega_1)}{i\omega_1+\gamma_p}\frac{U_\ell(\omega_2)}{i\omega_2+\gamma_p}\,. \qquad (3.43)\end{aligned}$$

Using Eq. (3.43) instead of Eq. (3.39) we obtain instead of Eq. (3.40)

$$\begin{aligned} v_{\ell,1}^2 &\equiv \langle [v_{\ell,1}(t)]^2\rangle = \int\frac{d\omega}{2\pi}F_{u,\ell,1}(\omega)\,, \qquad (3.44)\\ F_{u,\ell,1}(\omega) &= \frac{2\gamma_p^4}{(\omega^2+\gamma_p^2)\ell^2}\int\frac{d\omega_1 d\omega_2}{2\pi}\\ &\times\frac{\overline{F_{u,\ell}(\omega_1)}\;\overline{F_{u,\ell}(\omega_2)}\delta(\omega+\omega_1+\omega_2)}{(\omega_1^2+\gamma_p^2)(\omega_2^2+\gamma_p^2)}\,.\end{aligned}$$

In this derivation we assumed for simplicity the Gaussian statistics of the velocity field.

Now let us estimate

$$v_{\ell,1}^2 \approx \frac{\left[\overline{F_{u,\ell}(0)}\right]^2}{\ell^2} \approx \frac{u_\ell^4}{u_*^2} \approx u_\ell^2\left(\frac{\ell}{\ell_*}\right)^{2/3}\,, \qquad (3.45)$$

that is much larger than the result (3.42) for $v_{\ell,0}^2$ obtained in the linear approximation. This means that the simple iteration procedure we used is inconsistent, since it involves expansion in large parameter $[(\ell_*/\ell)^{1/3}]$.

e. Renormalized perturbative expansion. A similar situation with a perturbative expansion occurs in the theory of hydrodynamic turbulence, where a simple iteration of the nonlinear term with respect to the linear (viscous) term, yields the power series expansion in $\mathcal{R}e^2 \gg 1$. A way out, used in the theory of hydrodynamic turbulence is the Dyson-Wyld re-summation of one-eddy irreducible diagrams (for details see, e.g., Refs. [26, 27, 28]). This procedure corresponds to accounting for the nonlinear (so-called "turbulent" viscosity) instead of the linear, kinematic viscosity. A similar approach in our problem implies that we have to account for the self-consistent, nonlinear renormalization of the particle frequency $\gamma_p \Rightarrow \Gamma_p(\ell)$ in Eq. (3.38) and to subtract the corresponding terms from $\tilde{\mathcal{N}}_\omega$. With these corrections, Eq. (3.38) reads:

$$[\,i\omega + \Gamma_p(\ell)\,]V_\ell(\omega) = \gamma_p U_\ell(\omega) + \tilde{\mathcal{N}}_\omega\,. \qquad (3.46)$$

Here $\tilde{\mathcal{N}}_\omega$ is the nonlinear term $\mathcal{N}_\omega$ after substraction of the nonlinear contribution to the difference

$$\Delta_p \equiv \Gamma_p(\ell) - \gamma_p \approx \frac{v_\ell^2/\ell^2}{\Gamma_p(\ell)}\,. \qquad (3.47)$$

The latter relation actually follows from a more detailed perturbation diagrammatic approach. In our context it is sufficient to realize that in the limit $\Gamma_p(\ell) \gg \gamma_p$ one may evaluate $\Gamma_p(\ell)$ by a simple dimensional reasoning:

$$\Gamma_p(\ell) \approx v_\ell/\ell\,, \qquad (3.48)$$

which is consistent with Eq. (3.47). In addition, Eq. (3.47) has a natural limiting case $\Gamma_p(\ell) \to \gamma_p$ when $v_\ell/\ell \ll \gamma_p$. Now using Eq. (3.46) instead of Eq. (3.39) we arrive at:

$$V_\ell^{(0)}(\omega) = \frac{\gamma_p U_\ell(\omega)}{i\omega + \Gamma_p(\ell)}\,. \qquad (3.49)$$

Accordingly, instead of the estimates (3.42) one has:

$$\tilde{v}_{\ell,0}^2 \approx u_\ell^2\frac{\gamma_p^2\,\ell}{\Gamma_p(\ell)\,u_*} \approx u_\ell^2\frac{\gamma_p}{\Gamma_p(\ell)}\left(\frac{\ell}{\ell_*}\right)\,. \qquad (3.50)$$

The latter equation together with Eq. (3.48) allows to evaluate $\Gamma_p(\ell)$ as follows:

$$\Gamma_p(\ell) \approx \left(\frac{\gamma_p^2 u_\ell^2}{\ell\,u_*}\right)^{1/3} \approx \gamma_p\left(\frac{\ell_*}{\ell}\right)^{1/9}\,. \qquad (3.51)$$

Hence the estimate (3.50) becomes

$$\tilde{v}_{\ell,0}^2 \approx u_\ell^2 \left(\frac{\ell}{\ell_*}\right)^{10/9} \approx u_\ell^2 \left[\frac{\tau(\ell)}{\tau_p}\right]^{5/3} \, . \tag{3.52}$$

Repeating the evaluation of the nonlinear correction $\tilde{v}_{\ell,2}^2$ with the renormalized Eq. (3.46) we find that

$$\tilde{v}_{\ell,1}^2 \approx \tilde{v}_{\ell,0}^2 \, . \tag{3.53}$$

This means that now the expansion parameter is of the order of 1, in accordance with the renormalized perturbation approach.

IV. THE CLUSTERING INSTABILITY OF THE SECOND MOMENT OF PARTICLE NUMBER DENSITY

In this section we will present a quantitative analysis for the clustering instability of the second moment of particles number density.

A. Basic equations

To determine the growth rate of the clustering instability let us consider the equation for the two-point correlation function $\Phi(t, \boldsymbol{R})$ of particle number density:

$$\frac{\partial \Phi}{\partial t} = [B(\boldsymbol{R}) + 2\boldsymbol{U}(\boldsymbol{R}) \cdot \boldsymbol{\nabla} + \hat{D}_{\alpha\beta}(\boldsymbol{R}) \nabla_\alpha \nabla_\beta] \, \Phi(t, \boldsymbol{R}) \, , \tag{4.1}$$

(see Ref. [14]). The meaning of the coefficients $B(\boldsymbol{R})$, $\boldsymbol{U}(\boldsymbol{R})$ and $\hat{D}_{\alpha\beta}(\boldsymbol{R})$ is as follows:

The function $B(\boldsymbol{R})$ is determined by the compressibility of the velocity field and it causes the generation of fluctuations of the number density of particles.

The vector $\boldsymbol{U}(\boldsymbol{R})$ determines a scale-dependent drift velocity which describes a transport of fluctuations of particle number density from smaller scales to larger scales, i.e., in the regions with larger turbulent diffusion. The latter can decrease the growth rate of the clustering instability. Note that $\mathbf{U}(\boldsymbol{R} = 0) = 0$ whereas $B(\boldsymbol{R} = 0) \neq 0$. For incompressible velocity field $\mathbf{U}(\boldsymbol{R}) = 0$ and $B(\boldsymbol{R}) = 0$.

The scale-dependent tensor of turbulent diffusion $\hat{D}_{\alpha\beta}(\boldsymbol{R})$ is also affected by the compressibility. In very small scales this tensor is equal to the tensor of the molecular (Brownian) diffusion, while in the vicinity of the maximum scale of turbulent motions this tensor coincides with the regular tensor of turbulent diffusion.

Thus, the clustering instability is determined by the competition between these three processes. The tensor $\hat{D}_{\alpha\beta}(\boldsymbol{R})$ may be written as

$$\begin{aligned} \hat{D}_{\alpha\beta}(\boldsymbol{R}) &= 2D\delta_{\alpha\beta} + D^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R}) \, , \\ D^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R}) &= \tilde{D}^{\mathrm{T}}_{\alpha\beta}(0) - \tilde{D}^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R}) \, . \end{aligned} \tag{4.2}$$

The form of the coefficients $B(\boldsymbol{R})$, $\boldsymbol{U}(\boldsymbol{R})$ and $\hat{D}_{\alpha\beta}(\boldsymbol{R})$ depends on the model of turbulent velocity field. For instance, for the random velocity with Gaussian statistics of the Lagrangian trajectories $\boldsymbol{\xi}(\boldsymbol{r}|t)$ these coefficients are given by

$$\begin{aligned} B(\boldsymbol{R}) &\approx 2\int_0^\infty \langle b[0, \boldsymbol{\xi}(\boldsymbol{r}_1|0)] b[\tau, \boldsymbol{\xi}(\boldsymbol{r}_2|\tau)] \rangle \, d\tau \, , \\ \boldsymbol{U}(\boldsymbol{R}) &\approx -2\int_0^\infty \langle \boldsymbol{v}[0, \boldsymbol{\xi}(\boldsymbol{r}_1|0)] b[\tau, \boldsymbol{\xi}(\boldsymbol{r}_2|\tau)] \rangle \, d\tau \, , \\ \tilde{D}^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R}) &\approx 2\int_0^\infty \langle v_\alpha[0, \boldsymbol{\xi}(\boldsymbol{r}_1|0)] v_\beta[\tau, \boldsymbol{\xi}(\boldsymbol{r}_2|\tau)] \rangle \, d\tau \end{aligned} \tag{4.3}$$

(for details see Ref. [14]), where $b = \mathrm{div}\, \mathbf{v}$.

B. Clustering instability

Let us study the clustering instability. Consider the range of scales $a \leq \ell \ll \ell_*$, where the size of a particle $a \geq \eta$. Then the relationship between $\tilde{v}_{\ell,0}^2$ and u_ℓ^2 reads:

$$\tilde{v}_{\ell,0}^2 = u_\ell^2 \left[\frac{\tau(\ell)}{\tau_p}\right]^s \, , \tag{4.4}$$

where according to Eq. (3.52) the exponent $s = 5/3$. In this case the expression for the turbulent diffusion tensor in nondimensional form reads

$$\begin{aligned} D^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R}) &= R^{(4s-7)/3}(C_1 R^2 \delta_{\alpha\beta} + C_2 R_\alpha R_\beta) \, , \\ C_1 &= \frac{5 + 4s + 6\sigma_{\mathrm{T}}}{9\,(1 + \sigma_{\mathrm{T}})} \, , \\ C_2 &= \frac{(4s - 1)(2\sigma_{\mathrm{T}} - 1)}{9\,(1 + \sigma_{\mathrm{T}})} \, , \end{aligned} \tag{4.5}$$

where R is measured in the units of L and time t is measured in the units of $\tau_{L} \equiv \tau(\ell = L)$. The parameter σ_{T} is defined by analogy with Eq. (2.19):

$$\sigma_{\mathrm{T}} \equiv \frac{\boldsymbol{\nabla} \cdot \boldsymbol{D}_{\mathrm{T}} \cdot \boldsymbol{\nabla}}{\boldsymbol{\nabla} \times \boldsymbol{D}_{\mathrm{T}} \times \boldsymbol{\nabla}} = \frac{\nabla_\alpha \nabla_\beta D^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R})}{\nabla_\alpha \nabla_\beta D^{\mathrm{T}}_{\alpha'\beta'}(\boldsymbol{R}) \epsilon_{\alpha\alpha'\gamma} \epsilon_{\beta\beta'\gamma}} \, , \tag{4.6}$$

where $\epsilon_{\alpha\beta\gamma}$ is the fully antisymmetric unit tensor. Equations (2.19) and (4.6) imply that $\sigma_{\mathrm{T}} = \sigma_v$ in the case of δ-correlated in time compressible velocity field.

For a random incompressible velocity field with a finite correlation time the tensor of turbulent diffusion $\tilde{D}^{\mathrm{T}}_{\alpha\beta}(\boldsymbol{R}) = \tau^{-1} \langle \xi_\alpha(\boldsymbol{r}_1|t) \xi_\beta(\boldsymbol{r}_2|t) \rangle$ (see Ref. [14]) and the degree of compressibility of this tensor is

$$\sigma_{\mathrm{T}} = \frac{\langle (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 \rangle}{\langle (\boldsymbol{\nabla} \times \boldsymbol{\xi})^2 \rangle} \, , \tag{4.7}$$

where $\boldsymbol{\xi}(\boldsymbol{r}|t)$ is the Lagrangian trajectory.

To determine the functions $B(\boldsymbol{R})$ and $\boldsymbol{U}(\boldsymbol{R})$ we use the general form of the two-point correlation function

of the particle velocity field in the the range of scales $\eta \le \ell \ll \ell_*$:

$$\langle v_\alpha(t,\boldsymbol{r}) v_\beta(t + \tau, \boldsymbol{r}+\boldsymbol{R})\rangle = \frac{1}{3}[\delta_{\alpha\beta} - (C_1^v R^2 \delta_{\alpha\beta} + C_2^v R_\alpha R_\beta)\, R^{2(s-2)/3}] f(\tau) , \quad (4.8)$$

$$C_1^v = \frac{(4+s+3\sigma_v)}{3\,(1+\sigma_v)} ,$$

$$C_2^v = \frac{(1+s)(2\sigma_v - 1)}{3\,(1+\sigma_v)} .$$

Substitution Eq. (4.8) into Eq. (4.3) yields

$$\boldsymbol{U}(\boldsymbol{R}) = U_0\, R^{(4s-7)/3} , \quad (4.9)$$
$$B(\boldsymbol{R}) = B_0\, R^{(4s-7)/3} ,$$

where

$$U_0 = \beta\, \frac{\sigma_v}{\sigma_v + 1} , \quad B_0 = \alpha\, U_0$$

and the coefficients α and β depend on the properties of turbulent velocity field. The dimensionless functions B_0 and U_0 in Eq. (4.9) are measured in the units of $\tau_{_L}^{-1}$.

For the δ-correlated in time random Gaussian compressible velocity field $\sigma_{_{\rm T}} = \sigma_v$ and

$$B(\boldsymbol{R}) = \nabla_\alpha \nabla_\beta \hat D_{\alpha\beta}(\boldsymbol{R}) , \quad (4.10)$$
$$U_\alpha(\boldsymbol{R}) = \nabla_\beta \hat D_{\alpha\beta}(\boldsymbol{R}) \quad (4.11)$$

(for details see Ref. [14, 17, 18]). In this case the second moment $\Phi(t,\boldsymbol{R})$ can only decay, in spite of the compressibility of the velocity field.

For the finite correlation time of the turbulent velocity field $\sigma_{_{\rm T}} \neq \sigma_v$ and Eqs. (4.10) and (4.11) are not valid. The clustering instability depends on the ratio $\sigma_{_{\rm T}}/\sigma_v$. In order to provide the correct asymptotic behaviour of Eq. (4.9) in the limiting case of the δ-correlated in time random Gaussian compressible velocity field we have to choose the coefficients β and α in the form:

$$\beta = 8(4s^2 + 7s - 2)/27 , \; \alpha = (4s+2)/3 .$$

Note that when $s < 1/4$ the parameters $\beta < 0$ and $B(\boldsymbol{R}) < 0$. In this case there is no clustering instability of the second moment of particle number density.

Thus, Eq. (4.1) in a non-dimensional form reads:

$$\frac{\partial \Phi}{\partial t} = R^{(4s-7)/3}[R^2 \Phi''(C_1 + C_2) + 2\, R \Phi'(U_0 + C_1) + B_0 \Phi] . \quad (4.12)$$

Consider a solution of Eq. (4.12) in the vicinity of the thresholds of the excitation of the clustering instability, where $(\partial\Phi/\partial t) R^{(7-4s)/3}$ is very small. Thus, the solution of (4.12) in this region is

$$\Phi(R) = A_1 R^{-\lambda_1} , \quad (4.13)$$

where $\lambda_1 = \lambda \pm i\mu$,

$$\lambda = \frac{C_1 - C_2 + 2U_0}{2(C_1 + C_2)} , \quad \mu = \frac{C_3}{2(C_1+C_2)} ,$$
$$C_3^2 = 4B_0(C_1+C_2) - (C_1 - C_2 + 2U_0)^2 .$$

Since the correlation function $\Phi(R)$ has a global maximum at $R = a$, the coefficient $C_1 > C_2 - 2U_0$ if μ is a real number (see below). Thus the asymptotic solution of the equation for the two-point correlation function $\Phi(t,\boldsymbol{R})$ of the particle number density in the range of scales $a \le \ell \le \ell_*$ reads

$$\Phi(R) = A_1 R^{-\lambda} \sin[\mu \ln(\ell_*/R)] , \quad (4.14)$$

where

$$A_1 = \left(\frac{L}{a}\right)^\lambda \frac{1}{\sin[\mu \ln(\ell_*/a)]} .$$

Now consider the range of scales $\ell_* \ll \ell \ll L$. In this case the nondimensional form of the turbulent diffusion tensor is given by

$$D^{\rm T}_{\alpha\beta}(\boldsymbol{R}) = R^{-2/3}(\tilde C_1 R^2 \delta_{\alpha\beta} + \tilde C_2 R_\alpha R_\beta) , \quad (4.15)$$
$$\tilde C_1 = \frac{2(5+3\tilde\sigma_{_{\rm T}})}{9\,(1+\tilde\sigma_{_{\rm T}})} , \quad \tilde C_2 = \frac{4(2\tilde\sigma_{_{\rm T}} - 1)}{9\,(1+\tilde\sigma_{_{\rm T}})} ,$$

and Eq. (4.1) reads:

$$\frac{\partial \Phi}{\partial t} = R^{-2/3}[R^2 \Phi''(\tilde C_1 + \tilde C_2) + 2\, R \Phi' \tilde C_1] . \quad (4.16)$$

Here we took into account that in this range of scales the functions $B(\boldsymbol{R})$ and $\boldsymbol{U}(\boldsymbol{R})$ are negligibly small, and the degree of compressibility of the turbulent diffusion tensor can be different in the above two ranges of scales.

Consider a solution of Eq. (4.16) in the vicinity of the thresholds of the excitation of the clustering instability, when $(\partial\Phi/\partial t)R^{2/3}$ is very small. Thus, the solution of (4.16) is given by

$$\Phi(R) = A_2 R^{-\lambda_2} , \quad (4.17)$$

where

$$\lambda_2 = \frac{|\tilde C_1 - \tilde C_2|}{\tilde C_1 + \tilde C_2} = \frac{|7 - \tilde\sigma_{_{\rm T}}|}{3 + 7\tilde\sigma_{_{\rm T}}} .$$

The growth rate of the second moment of particle number density can be obtained by matching the correlation function $\Phi(R)$ and its first derivative $\Phi'(R)$ at the boundary of the above two ranges of scales, i.e., at the points $R = \ell_*/L$. The matching yields that in the range of scales $\ell_* \ll \ell \le L$ the asymptotic solution of the equation for the two-point correlation function $\Phi(t,\boldsymbol{R})$ has the form of Eq. (4.17) with

$$A_2 = (-1)^k A_1 \left(\frac{\ell_*}{L}\right)^{\lambda_2 - \lambda + 1} \frac{\mu}{\lambda_2} .$$

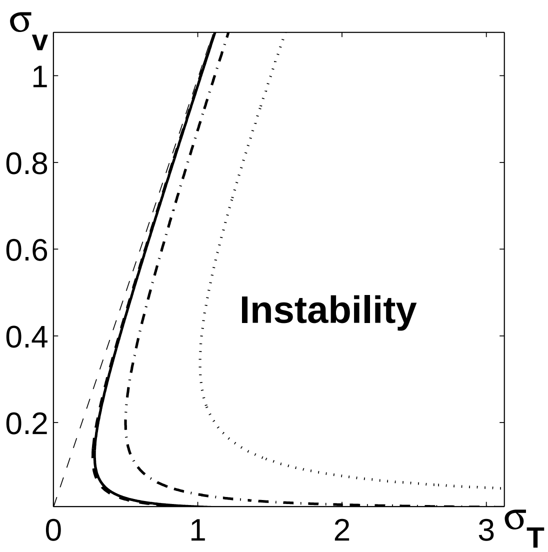

FIG. 1: The range of parameters $(\sigma_v, \sigma_{\rm T})$ for which the clustering instability may occur. The various curves indicate results for $s = 7/4$ (dashed), $s = 5/3$ (solid), for $s = 1$ (dashed-dotted) and for $s = 2/3$ (dotted). The thin dashed line $\sigma_v = \sigma_{\rm T}$ corresponds to the δ-correlated in time random compressible velocity field.

Such matching is possible only when λ_1 is a complex number, i.e., when $C_3^2 > 0$ (i.e., μ is a real number). The latter determines the necessary condition for the clustering instability of particle spatial distribution. The range of parameters $(\sigma_v, \sigma_{\rm T})$ for which the clustering instability of the second moment of particle number density may occur is shown in Fig. 1. The line $\sigma_v = \sigma_{\rm T}$ corresponds to the δ-correlated in time random compressible velocity field for which the clustering instability cannot be excited. The various curves indicate results for different value of the parameter s. In particular, the value $s = 7/4$ corresponds to the turbulent diffusion tensor with the scaling $\propto R^2$ [see Eq. (4.5)]. The curves for $s = 7/4$ (dashed) and $s = 5/3$ (solid) practically coincide. The parameter s can be considered as a phenomenological parameter, and the change of this parameter from $s = 5/3$ to $s = 0$ can describe a transition from one asymptotic behaviour (in the range of scales $a \leq \ell \leq \ell_*$) to the other ($\ell_* \ll \ell \leq L$).

V. DISCUSSION

Formation and evolution of particle clusters are of fundamental significance in many areas of environmental sciences, physics of the atmosphere and meteorology (e.g., smog and fog formation, rain formation), see e.g., Ref. [29, 30, 31, 32], transport and mixing in industrial turbulent flows (like spray drying and cyclone dust separation, dynamics of fuel droplets), see e.g., Ref. [33, 34, 35, 36, 37, 38] and references therein. The analysis of the experimental data showed that the spatial distributions of droplets in clouds are strongly inhomogeneous [12]. The small-scale inhomogeneities in particle distribution were observed also in laboratory turbulent flows [10, 11].

The analyzed effect of particle clustering may be of relevance in turbulent fluid flows of different nature with inertial particles or droplets (e.g., in atmospheric turbulence, combustion and in a laboratory turbulence). In particular, this effect can cause formation of small-scale inhomogeneities in spatial distribution of fuel droplets in diesel engines. The starting point is the above theoretical model that describes the inertial particle clustering. For numerical estimates we adopt the operating parameters for a typical diesel engines, taking the crankshaft rotational speed N=3000 rev/min, that corresponds to the average piston speed, $\bar{v}_p = 10^3$ cm/s, and the turbulent velocity $v' \simeq 5 \cdot 10^2$ cm/s. Taking into account that the spatial integral space (the characteristic scale of the largest turbulent eddies) for this conditions is $L \simeq 0.3$ cm (see Ref. [36]), we find for the Reynolds number:

$$\mathcal{R}e \;=\; \frac{v' L}{\nu} \simeq 10^4\,, \quad \text{with} \tag{5.1}$$
$$v' \;\simeq\; 5 \cdot 10^2 \text{cm/sec.}, \quad L \simeq 0.3\,\text{cm}\;.$$

Here the kinematic viscosity of fluid is $\nu \approx 0.015\,\text{cm}^2/\text{s}$. To estimate ν we used the following values for the fluid density $\rho = 0.025\,\text{g/cm}^3$ (at 62 atm. and T=860 K), and for kinematic viscosity

$$\nu = \nu_0 R^{(3-\gamma)/2}, \tag{5.2}$$

where $R = (\rho/\rho_0) \approx 15$ is the compression ratio. Then the Kolmogorov micro-scale is $\eta \approx L/\mathcal{R}e^{3/4} \sim 3\,\mu$m. A more accurate estimate is

$$\eta \approx \frac{L}{(\mathcal{R}e/\mathcal{R}e_{\rm cr})^{3/4}}\,, \tag{5.3}$$

where $\mathcal{R}e_{\rm cr} > 1$ is the critical value of $\mathcal{R}e$ at which the laminar flow becomes unstable. Taking $\mathcal{R}e_{\rm cr} \approx 5$, we obtain a more realistic value $\eta \simeq 10\,\mu$m.

Note that turbulence in the in diesel engines is neither homogeneous nor isotropic. However, we study dynamics of droplets at very small scales which are of the order of tens of hundreds microns. In such small scales the turbulence can be considered as a quasi isotropic and homogeneous. Anisotropy and inhomogeneities of turbulence can be essential in scales which are of the order of 1 cm and larger.

Consider a region of unstable cluster scales: $\eta < \ell < \ell_*$. Assuming the fuel density in droplets $\rho_{\rm dr} = 0.85\,\text{g/cm}^3$, we find from the expression (3.18) the estimate for the critical value of the droplet radius, $a_* \approx 4\,\mu$m, indicating that all droplets with $a > 4\,\mu$m are unstable against formation of the clusters with scales ranging from η to ℓ_*. This equation yields the following values of ℓ_*: $\ell_* \approx 160\,\mu$m for $a = 10\,\mu$m and $\ell_* \approx 1$cm for $a = 40\,\mu$m.

The characteristic time of the cluster growth $\tau_{cl}(\ell) = 1/\gamma_{cl}$ depends on the cluster size ℓ. Estimate for $\ell = \ell_*/3$ gives $\tau_{\rm cl}(\ell_*/3) \approx 1.7\tau(\ell_*) \approx \tau_{\rm dr}$. The turnover time $\tau(\ell)$ can be estimated as follows:

$$\tau(\ell) \simeq 5 \left(\frac{L}{v'}\right)\left(\frac{\ell}{L}\right)^{2/3} \simeq 3\times 10^{-3}\left(\frac{\ell}{L}\right)^{2/3} \text{sec} \; . \quad (5.4)$$

Here we used our estimates (5.1) for the integral scale L and turbulent velocity v'. Assuming for the droplets with the size $a = 10\,\mu$m, $\ell = \ell_*/3 \approx 50\,\mu$m (with $\ell_* = 160\,\mu$m) one has $\tau_{\rm cl}(\ell) \approx 0.35\,\mu$s. For the droplets with the radius $a = 40\,\mu$m with $\ell_* \approx 1$ cm, one has $\tau_{\rm cl}(\ell_*/3) \approx 5.5\,\mu$s, which is larger than the characteristic spray time in the combustion chamber (a few milliseconds). These estimates imply that for the chosen parameters the distribution of the droplets in the spray will be shifted to the droplets with larger diameters at the given turbulent intensity.

Now we estimate the time of the turbulent air-fuel mixing, $\tau_{\rm T} \simeq 1/\gamma_{\rm T}$ in the combustion chamber. Taking into account that

$$\gamma_{\rm T} = \frac{D_{\rm T}}{d^2} = \frac{v'L}{3d^2} \, , \quad (5.5)$$

where $D_{\rm T}$ is the coefficient of turbulent diffusion, and $d \simeq 10$ cm is the bore diameter, we find for the time of turbulent air-fuel mixing $\tau_{\rm T} \approx 1\,$s. Thus, the turbulent diffusion itself is too slow and a strong external global flow - "swirl" is needed for the effective air-fuel mixing.

In the previous sections we have shown that the particle spatial distribution in the turbulent flow field is unstable against formation of clusters with particle number density that is much higher than the average particle number density. Obviously this exponential growth at the linear stage of instability should be saturated by nonlinear effects. The nonlinear saturation for the water droplets in the atmospheric turbulence (in clouds) was discussed in Ref. [14]. Here we discuss a similar issue for the fuel droplets under the typical conditions pertinent to diesel engines.

A momentum coupling of particles and turbulent fluid is essential when $m_p n_{\rm cl} \sim \rho$, i.e., the mass loading parameter, $\phi = m_p n_{\rm cl}/\rho$, is of the order of unity (see, e.g., Ref. [39]). This condition implies that the kinetic energy of air $\rho\langle {\bf u}^2\rangle$ is of the order of the droplets kinetic energy $m_p n_{\rm cl}\langle {\bf v}^2\rangle$, where $|{\bf u}| \sim |{\bf v}|$. This yields:

$$n_{\rm cl} \sim a^{-3}(\rho/3\rho_p) \; . \quad (5.6)$$

For the fuel droplets in the diesel combustion chambers (see, e.g. Ref. [34]) $\rho_p/\rho \simeq 34$. Thus, e.g., for $a \simeq 10\,\mu$m we obtain $n_{\rm cl} \sim 3\times 10^7$ cm^{-3}. For the cluster with the size $\ell \approx \ell_*/3 \approx 50\,\mu$m this yields for the total number of particles in the cluster of that size $N_{\rm cl} \simeq \ell^3 n_{\rm cl} \sim 30$. Note that the mean number density of droplets in a combustion camera $\bar n$ is about 10^4 cm^{-3}. Therefore the clustering instability of droplets in the diesel engines increases their concentrations in the clusters by the orders of magnitude.

Note that in the initial stage of the clustering instability, when the mass loading parameter is small, i.e., for the small number density of droplets, we can neglect the droplets collisions, and consider only the collisions between the droplets and the molecules of the surrounding fluid (air). In the nonlinear stage of the clustering instability, the four-way coupling can be effective, and the kinetic approach for analysis of the droplets collisions inside the cluster can be important. Moreover, the coagulations of droplets due to their collisions change the size of the droplets. This effect is determined by the kinetic Smoluchovsky equation for droplets size distribution. However, the interaction between droplets and the surrounding fluid even at this stage of the clustering instability can be described on the level of the continuum hydrodynamic approach. The main purpose of the present paper is to describe the initial (linear) stage of the clustering instability and to determine the conditions for the onset of the clustering instability of fuel droplets. This stage of the instability can be analyzed only using the continuum hydrodynamic approach because the mean free path l_c of the molecules of the surrounding fluid is much smaller than all the characteristic spatial scales in the problem (e.g., the Kolmogorov micro-scale η, the size of droplets a, etc). Indeed, for diesel engines these parameters are: $l_c = 0.01\,\mu$m, $\eta = 3\,\mu$m, $a \geq 10\,\mu$m. Similarly, the mean time τ_c between collisions of the molecules of the surrounding fluid is much smaller than all the characteristic time scales (e.g., the correlation time in the Kolmogorov micro-scale τ_η, the Stokes time for droplets τ_p, etc). Indeed, for diesel engines these parameters are $\tau_c = 2\times 10^{-11}\,$s, $\tau_\eta = 3\times 10^{-5}\,$s, and $\tau_p = 10^{-4}\,$s for $a = 10\,\mu$m. At the strongly nonlinear stage of the clustering instability, when the mean free path of the droplets inside a cluster becomes much smaller than the size of the cluster, the continuum hydrodynamic approach can be still used to obtain a rough estimate of the number density of droplets inside the cluster at the saturation of the clustering instability.

Acknowledgments

This research was supported in part by The German-Israeli Project Cooperation (DIP) administrated by the Federal Ministry of Education and Research (BMBF), by the Israel Science Foundation governed by the Israeli Academy of Science, by Swedish Ministry of Industry (Energimyndigheten, contract P 12503-1), by the Swedish Royal Academy of Sciences, the STINT Fellowship program.

[1] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence* (M.I.T. Press, Cambridge, 1975), vol. 2.
[2] G.T. Csanady, *Turbulent Diffusion in the Environment* (Reidel, Dordrecht, 1980).
[3] F. Pasquill and F.B. Smith, *Atmospheric Diffusion* (Ellis Horwood, Chichester, 1983).
[4] W.D. McComb, *The Physics of Fluid Turbulence* (Clarendon Press, Oxford, 1990).
[5] D. Stock, *J.* Fluids Engineering **118**, 4 (1996).
[6] A.K. Blackadar, *Turbulence and Diffusion in the Atmosphere* (Springer, Berlin, 1997).
[7] Z. Warhaft, Annu. Rev. Fluid Mech. **32**, 203 (2000).
[8] B. Sawford, Annu. Rev. Fluid Mech. **33**, 289 (2001).
[9] R.E. Britter and S.R. Hanna, Annu. Rev. Fluid Mech. **35**, 469 (2003).
[10] J.R. Fessler, J.D. Kulick and J.K. Eaton, Phys. Fluids **6**, 3742 (1994).
[11] A. Aliseda, A. Cartellier, F. Hainaux and J.C. Lasheras, J. Fluid Mech. **468**, 77 (2002).
[12] R.A. Shaw, Ann. Rev. Fluid Mech. **35**, 183 (2003).
[13] T. Elperin, N. Kleeorin and I. Rogachevskii, Phys. Rev. Lett. **77**, 5373 (1996).
[14] T. Elperin, N. Kleeorin, V. L'vov, I. Rogachevskii and D. Sokoloff, Phys. Rev. E **66**, 036302 (2002). Also e-prints: nlin.CD/0202048 and nlin.CD/0204022.
[15] T. Elperin, N. Kleeorin and I. Rogachevskii, Phys. Rev. Lett. **76**, 224 (1996).
[16] T. Elperin, N. Kleeorin and I. Rogachevskii, Phys. Rev. Lett. **81**, 2898 (1998).
[17] T. Elperin, N. Kleeorin, I. Rogachevskii and D. Sokoloff, Phys. Chem. Earth, **A 25**, 797 (2000).
[18] T. Elperin, N. Kleeorin, I. Rogachevskii and D. Sokoloff, Phys. Rev. E **63**, 046305 (2001).
[19] R.H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[20] M.R. Maxey and J.J. Riley, Phys. Fluids **26**, 883 (1983).
[21] M.R. Maxey, J. Fluid Mech. **174**, 441 (1987).
[22] M.R. Maxey, E.J. Chang, and L.-P. Wang, Experim. Thermal and Fluid Science, **12**, 417 (1996).
[23] L.D. Landau and E.M. Lifshits, *Fluid mechanics* (Pergamon, Oxford, 1987).
[24] U. Frisch, *Turbulence: The Legasy of A. N. Kolmogorov* (Cambridge University Press, Cambridge 1995).
[25] J.L. Lumley, in *T*wo-Phase and Non-Newtonian Flows, Turbulence, ed. P. Bradshaw, (Springer Verlag, Berlin, 1976).
[26] V.S. L'vov and I. Procaccia, in *Lecture Notes of the Les Houches Summer School "Fluctuating Geometries in Statistical Mechanics and Field Theory"*, ed. by F. David and P. Ginsparg, (North-Holland, Amsterdam, 1995).
[27] V.S. L'vov and I. Procaccia, Phys. Rev. E **52**, 3840 (1995).
[28] V.S. L'vov and I. Procaccia, Phys. Rev. E **52**, 3858 (1995).
[29] J.H. Seinfeld, *A*tmospheric Chemistry and Physics of Air Pollution, (John Wiley, New York, 1986), and references therein.
[30] R. Flagan and J.H. Seinfeld, *F*undamentals of air pollution engineering, (Prentice Hall, Englewood Cliffs, 1988).
[31] H.R. Pruppacher and J.D. Klett, *Microphysics of Clouds and Precipitation*, (Kluwer Acad. Publ., Dordrecht, 1997).
[32] P.A. Vaillancourt and M.K. Yau, Bull. Americ. Meteorol. Soc. **81**, 285 (2000), and references therein.
[33] C.T. Crowe, M. Sommerfeld and Y. Tsuji, *M*ultiphase Flows with Particles and Droplets (CRC Press, NY, 1998).
[34] J.B. Heywood, *I*nternal Combustion Engine Fundamentals, (MacGraw-Hill, Boston-New-York, 1988).
[35] T. Kamimoto and H. Kobayashi, Prog. Energy Combust. Sci. **17**, 163 (1991).
[36] G.L. Borman and K.W. Ragland, *C*ombustion Engineering, (MacGraw-Hill, Boston-New-York, 1999).
[37] J. Bellan, Prog. Energy and Combust. Sci. **28**, 329 (2000).
[38] H.H. Chiu, Prog. Energy and Combust. Sci. **26**, 381 (2000).
[39] V.S. L'vov, G. Ooms and A. Pomyalov, "Effect of particle inertia on the turbulence in suspensions," Phys. Rev. E **67**, 046314 (2003). Also e-print: nlin.CD/0210069.